$g \to$ affine algebra

$a_\lambda^\Lambda \quad \mathrm{ch}_{L(\Lambda)} = \sum_{\substack{\lambda \in \max(\Lambda) \\ \lambda \bmod T}} \left(\sum_{t \in T} e^{t(\lambda)} \right) a_\lambda^\Lambda \longrightarrow c_\lambda^\Lambda$: string function

$(\lambda \in \max(\Lambda))$

$\lambda \in \mathfrak{h}^* \longleftrightarrow \chi_\Lambda = \sum_{\substack{\lambda \in P_+^k \bmod \mathbb{C}\delta \\ \lambda \bmod T}} c_\lambda^\Lambda \Theta_\lambda$

branching function $\leftarrow$ D. H. Peterson. [24. 26]

(generalized string function)

$\leftarrow$ Sugawara [37. 26. 13] conformal invariance on braching functions

Sugawara operator

$\hat{L}(\dot{g})$

$d_\lambda^\Lambda \to$ braching function

$\chi_\Lambda = e^{-m_\Lambda\delta} \mathrm{ch}_{L(\Lambda)} = \sum_{\substack{\lambda \in P_+^k \bmod \mathbb{C}\delta \\ \lambda \bmod T}} c_\lambda^\Lambda \Theta_\lambda$

normalized character

Recall. $P_+ = \{ \lambda \in \dot{\mathfrak{h}}^* \mid \lambda|_{\dot{\mathfrak{h}} \cap \hat{L}(\dot{g}_{(i)})} \in P_{+(i)}$ for $i \geq 1$ and

$\lambda|_{\dot{\mathfrak{h}} \cap g_{(0)\mathbb{R}}} \in \mathbb{R}$ and $\langle \lambda, k_0 \rangle > 0 \}$

$P_+^k = \{ \lambda \in P_+ \mid \lambda(k_i) = k_i \ (i = 0, 1 \cdots) \}$

Fix $\Lambda \in P_+$ $\quad L(\Lambda) \cong \bigoplus \dot{L}(\lambda)$, where $\lambda \in \dot{P}_+ + \mathbb{C}\delta$ [Kac-Peterson Adv. 1984 Sec 4g]

as $\hat{L}(\dot{g})$ module

$\mathrm{mult}_\Lambda(\lambda, \dot{g}) = \mathrm{mult}\,\dot{L}(\lambda)$ — $\dot{g}$ 不可约的个数

the multiplicity of occurrence of $\dot{L}(\lambda)$ in $L(\Lambda)$

Rmk: $\mathrm{mult}_\Lambda(\lambda, \dot{g}) = \dim L(\Lambda)_\lambda^{\dot{n}_+}$

Rmk: $\mathrm{mult}_\Lambda(\lambda, \dot{g}) = \dim \{ v \in L(\Lambda) \mid h(v) = \lambda(h) \cdot v$ for $h \in \dot{\mathfrak{h}}$, $\dot{n}_+(v) = 0 \}$ $\quad \{*\}$

$\cancel{\mathrm{mult}}\ L(\Lambda)_\lambda^{\dot{n}_+}$ $\quad$ 分量的形式

$L(\Lambda)_\lambda^{\dot{n}_+}$ $\Rightarrow c_\lambda^\Lambda = b_\lambda^\Lambda(\dot{g}) / \eta^L$

数这样的 v 有多少个

aim: studying braching function

$\chi_\Lambda = \sum_{\lambda \in P_+^k \bmod (\mathbb{C}\delta + k\Omega M)} b_\lambda^\Lambda(\dot{g}) \dot{\chi}_{\lambda(1)} (\dot{\theta}^0_{\lambda(0)} / \dot{\eta}^{l_0})$

$\dot{g}$ 取好.

$\chi_\Lambda = \sum_{\lambda \in P_+^k \bmod (\mathbb{C}\delta)} b_\lambda^\Lambda \theta_{\lambda(0)} / \eta^{l_0}$

$$b_\lambda^\Lambda(\dot{g}) = e^{-\delta(m_\Lambda - \dot{m}_\lambda)} \sum_{n \in \mathbb{Z}} \mathrm{mult}_\Lambda(\lambda - n\delta, \dot{g})\, e^{-n\delta}$$

§12.1

$g = n_- \oplus \mathfrak{h} \oplus n_+$

$\dot{g} = \dot{n}_+ \cdots$

$\hat{L}(\dot{g}) = n_- \oplus \mathfrak{h} \oplus n_+$

$\hat{L}(\dot{g}) = \dot{n}_- + \dot{\mathfrak{h}} + \dot{n}_+$

$$h_\Lambda = \frac{(\Lambda + 2\rho | \Lambda)}{2(k + h^\vee)} \qquad \dot{h}_\lambda = \sum_i \frac{(\lambda_i + 2\dot{\rho}_i | \lambda_i)}{2(k_i + h_i^\vee)}$$

Defi: (Vacuum pair of level k)

The pair $M \in P_+^k$ and $u \in P(M)|_{\dot{\mathfrak{h}}} \cap \dot{P}_+^k$ 限制 such that $h_M = \dot{h}_u$ is called a vacuum pair of level k 真空对. denote by $R_k = \{ (M; u) \mid \cdots \}$

Prop 12.1 Let $(M; u) \in R_k$, then $\mathrm{mult}_M(u; \dot{g}) = \sum_{\substack{\bar{u} \in P(M) \\ \bar{u}|_{\dot{\mathfrak{h}}} = u}} \mathrm{mult}_{L(M)}(\bar{u}) > 0$

Pf: Let $\bar{u} \in P(M)$ st $\bar{u}|_{\dot{\mathfrak{h}}} = u$. $\Rightarrow$ $L(M)_{\bar{u}} \subseteq \{*\}$

i.e $\forall v \neq 0 \in L(M)_{\bar{u}}$, we have to show that $\dot{n}_+(v) = 0$

In the contrary case, there exists $\beta \in \dot{Q}_+ \setminus \{0\}$ s.t $\mathrm{mult}_M(u + \beta; \dot{g}) > 0$. $L(u+\beta)$ $u+\beta$ 也出现

But then $\dot{h}_{u+\beta} - \dot{h}_u = \sum_i \frac{(u_i + \beta_i + 2\dot{\rho}_{(i)} | u_i + \beta_i) - (u_i + 2\dot{\rho}_{(i)} | u_i)}{2(k_i + h_i^\vee)}$

$$= \sum_i \frac{\left(\underset{\text{本权}}{(u_i + \beta_{(i)})} + u_{(i)} + 2\dot{\rho}_i \,\middle|\, \beta_{(i)} \right)}{2(k_i + h_i^\vee)} > 0$$

$\beta \in \dot{Q}_+ \setminus \{0\}$ 由 $(\cdot,\cdot) > 0$ $\quad$ $(\beta_{(i)} | \beta_{(i)}) > 0$ 正权

$\Rightarrow \underset{h_M}{h_u} < h_{u+\beta}$ $\quad$ $\left\{ \mathop{\text{prop 12.12b}}^{\mathrm{mult}(\lambda, \dot{g}) \neq 0} \underset{(M \in P_+)}{h_M \geq h_\lambda} \right.$ $\quad \Rightarrow \mathrm{mult}_M(u; \dot{g}) = \sum_{\substack{\bar{u} \in P(M) \\ \bar{u}|_{\dot{\mathfrak{h}}} = u}} \mathrm{mult}_{L(M)}(\bar{u}) > 0$

②

in particular. $(M,u)\in R_K \Leftrightarrow h_M = \dot{h}_u$ and $\mathrm{mult}_M(u,\dot{g}) \neq 0$

$h_M = \dot{h}_u$ ~~and~~ $\mathrm{mult}_M(u,\dot{g}) \neq 0$

$u \in P(M)|_{\dot{g}} \cap \dot{P}_+^{k'}$, where $M \in P_+^k$

Rmk: ① $M\in P_+$, 无 $\dot{P}_+$ Since the module $L(M)$ is completely reducible with respect to $\hat{L}(\dot{g})$

we see from (12.9.5) $\boxed{L_0^{g} = \sum_i \frac{\Omega_i}{2(k_i + h_i^\vee)} - d}$ that $\boxed{L_0^{g,\dot{g}}}$ is a diagonalizable on $L(M)$

Ω_i commuts with $g_{(i)}$ 标量作用 $\Omega_i(v) =$ 可对角 $(\lambda_i + 2\rho | \lambda_i) v$ if $n_{\gamma}(\lambda) = 0$

$L_0^{g}(v) = (h_\lambda I - d) v$

$L_n^{g,\dot{g}}$ commute with $\hat{L}(\dot{g})$

from proof of prop 12.12b $\Rightarrow$ its spectre is non-negative. Each of its eigenspace is $\hat{L}(\dot{g})$ invariant (12.10.1)

~~Finally~~ Each ~~of~~ its zero eigenspace (the vacuum space) decomposes into a direct sum of $\hat{L}(\dot{g})$

modules $\dot{L}(u)$ st. $(M;u)$ is a ~~Vacuur~~ Vacuum pair, with $\mathrm{mult}(u;\dot{g})$

(这里和前面比多了 level 要求).

§12.12.

Def: For $\lambda \in P_+^k$ and $\lambda \in \dot{g}^*$, let

majorized by $\sum_{n\in\mathbb{Z}} \dim(L(\lambda))_{\lambda - n\delta} e^{-n\delta}$

$\ulcorner$ Its completely determined by $\sum_{j\geq 0} ~~(\dim L(\lambda)\ldots)~~ e^{-j\delta}$

$$b_\lambda^\Lambda(\dot{g}) = e^{-(m_\Lambda - \dot{m}_\lambda)\delta} \sum_{n\in\mathbb{Z}} \mathrm{mult}_\Lambda(\lambda - n\delta; \dot{g}) e^{-n\delta}$$

this series converges absolutely to a holomorphic function on $Y \to \mathfrak{h}_+$ 上半平面

called a branching function. (分支) $\mathbb{C}$-span of $\{b_\lambda^\Lambda \mid \Lambda \in P_+^k, \lambda \in \dot{g}^*\}$ is invariant under $SL_2(\mathbb{Z})$

↓ modular function

Rmk: string functions are special cases of branching functions:

(12.12.1). $c_\lambda^\Lambda = b_\lambda^\Lambda(\dot{\mathfrak{h}}) \eta^{-l}$ $\dot{g} \to$ 半单李代数 to 可约代数. abeian Lie algebra

$\ulcorner$ this follow from (12.8.12) $\chi_\lambda := e^{-m_\lambda\delta} \mathrm{ch}L(\lambda) = e^{-\frac{|\lambda|^2}{2k}\delta + \lambda} / \eta^{\dim \dot{g}}$

$$\eta = e^{-\frac{\delta}{24}} \varphi(e^{-\delta}) = e^{-\frac{\delta}{24}} \prod_{n=1}^{\infty} (1 - e^{-\delta n})$$

$\chi_\lambda = c_\lambda^\Lambda$

$\dim \dot{g} = \dim \dot{\mathfrak{h}} = l$

$c(k) = l$

$m_\lambda = \frac{|\lambda|^2}{24} - \frac{l}{24}$

$$m_\lambda = h_\lambda - \frac{1}{24} c(k) = h_\lambda - \frac{\dim \dot{\mathfrak{h}}}{24} = h_\lambda - \frac{l}{24}$$

$h_\lambda = \frac{|\lambda|^2}{24}$ $k \neq 0$

$h_\lambda = \frac{(\lambda|\lambda)}{2k}$

$\dot{\mathfrak{h}}$ abeliance

$L(\lambda) \leftarrow g = \hat{L}(\dot{g})$

$h^\vee = 0$

$\rho = 0$

$h^\vee = 1$

$M = 0$

$$\chi_\Lambda = \sum_{\lambda \in P^k \bmod (kM + \mathbb{C}\delta)} c_\lambda^\Lambda \theta_\lambda$$

$k = \mathrm{level}(\Lambda) \neq 0$

$\chi_\lambda = e^{m_\lambda\delta} \mathrm{ch}L(\lambda) = e^{-\frac{|\lambda|^2}{2k}\delta + \lambda} / \eta^l$

$\mathrm{ch}L(\lambda) = e^\lambda / \varphi(e^{-\delta})^l$

$= e^{k\Lambda_0} e$

$$\theta_\lambda = e^{k\Lambda_0} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{-\frac{1}{2}k|\gamma|^2\delta + k\gamma} e^{-m_\lambda\delta} =$$

$\theta_\lambda = e^{k\Lambda_0} \cdot e^{-\frac{1}{2}k|\frac{\bar{\lambda}}{k}|^2\delta + \frac{\bar{\lambda}}{k}}$

$l = \dim \dot{\mathfrak{h}}$

$$m_\lambda = h_\lambda - \frac{1}{24}c(k) = \frac{(\lambda|\lambda)}{2k} - \frac{1}{24} c(k)$$

[3]

The branching function have a simple representation theoretical meaning.

To explain this, let

$U(\Lambda,\lambda) = \{ v \in L(\Lambda) \mid \dot{n}_+(v) = 0$ and $h(v) = \langle \lambda, h \rangle v$ for $h \in \dot{\mathfrak{g}} \}$

这里有些不太.

$$\begin{cases} \mathcal{L}(\dot{\mathfrak{g}}) = \hat{n}_+ \oplus \dot{\mathfrak{g}} + \hat{n}_+ \\ \dot{\mathfrak{g}} = \dot{n}_- \oplus \dot{\mathfrak{h}} \oplus \dot{n}_+ \end{cases}$$

$\operatorname{mult}_M(u,\dot{\mathfrak{g}}) = \dim\{ v \in L(\Lambda) \mid h \cdot u = \lambda(h) u$ for $h \in \dot{\mathfrak{g}}, \dot{n}_+(v) = 0 \}$

这里有些不太.

$\| \ L(\Lambda)^{\dot{n}_+}_u$

Fact: $U(\Lambda,\lambda) = \bigoplus_{n \in \mathbb{Z}_+} L(\Lambda)^{\hat{n}_+}_{\lambda - n\delta}$

For $\lambda \in \max(\Lambda, \hat{\mathcal{L}}(\dot{\mathfrak{g}}))$
$\begin{cases} \operatorname{mult}_\Lambda(\lambda, \dot{\mathfrak{g}}) \neq 0 \\ \operatorname{mult}_\Lambda(\lambda + m\delta\, p) = 0 \quad \text{for } m > 0 \end{cases}$

[P. Goddard 1985]

Fact2: the subspace $U(\Lambda,\lambda)$ is a coset Vir-submodule with $c = c(k) - \dot{c}(k)$

∵ due to prop 12.10, with respect to V be a restricted $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$-module ($V = L(\Lambda)$)

$d_n \longmapsto L_n^{\mathfrak{g},\dot{\mathfrak{g}}}$ $\quad c \longrightarrow c(k) - \dot{c}(k)$

$\forall x \in \dot{\mathfrak{g}}\ m,n \in \mathbb{Z}$ $\quad [x^{(m)}, L_n^{\mathfrak{g}}] = m x^{m+n}$ $\quad [x^{(m)}, L_n^{\dot{\mathfrak{g}}}] = m x^{m+n}$

$[x^{(m)}, L_n^{\mathfrak{g},\dot{\mathfrak{g}}}] = 0$ $\quad [L_m^{\dot{\mathfrak{g}}}, L_n^{\mathfrak{g},\dot{\mathfrak{g}}}] = 0$ for $m,n \in \mathbb{Z}$

$\Rightarrow [L_m^{\mathfrak{g},\dot{\mathfrak{g}}}, L_n^{\mathfrak{g},\dot{\mathfrak{g}}}] = [L_m^{\mathfrak{g}}, L_n^{\mathfrak{g}}] - [L_m^{\dot{\mathfrak{g}}}, L_n^{\dot{\mathfrak{g}}}]$

$L_m^{\mathfrak{g},\dot{\mathfrak{g}}}(L(\Lambda)_\lambda) \subseteq$ ~~L~~ $L(\Lambda)_{\lambda + m\delta}$ for $\lambda \in \mathfrak{h}^*$, $m \in \mathbb{Z}$

$[x^{(m)}, L_n^{\mathfrak{g}\dot{\mathfrak{g}}}] = 0.$

$\Rightarrow L_m^{\mathfrak{g}\dot{\mathfrak{g}}}(U(\Lambda,\lambda)) \subset U(\Lambda,\lambda)$

Fact3: We have obtained the following decomposition of $L(\Lambda)$ with respect to $\hat{\mathcal{L}}(\dot{\mathfrak{g}}) \oplus \mathrm{Vir}$:

$$\boxed{\begin{aligned} L(\Lambda) &= \bigoplus_{\lambda \in \dot{P}_+^k \bmod \mathbb{C}\delta} \dot{L}(\lambda) \otimes U(\Lambda,\lambda) \\ \text{or}\quad L(\Lambda) &= \bigoplus_{\lambda \in \max(\Lambda, \dot{\mathfrak{g}})} \dot{L}(\lambda) \otimes U(\Lambda,\lambda) \end{aligned}}$$

1. by the complete reducibility theorem, with resp $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$
2. Similar argument as in prop 11.9. $\alpha \in \Delta_+^{im}$, $\Lambda \in P_+$.

By $U(\Lambda,\lambda)$ 定义

$$L(\Lambda) = \underline{L(\Lambda)_0^{(\alpha)}} \oplus (L(\Lambda)_+^{(\alpha)}) \quad \underline{= \bigoplus_{\lambda: (\lambda|\alpha) = 0} L(\Lambda)_\lambda} \oplus \left(\bigoplus_{\lambda: (\lambda|\alpha) > 0} L(\Lambda)_\lambda \right)$$

$L(\Lambda)_0^{(\alpha)} = \{ x \in L(\Lambda) \mid g^{(\alpha)}(x) = 0 \}$

$L(\Lambda)_+^{(\alpha)} = U(n_-^{(\alpha)}) \{ x \mid x \in L(\Lambda)_+^{(\alpha)} \mid n_+^{(\alpha)}(x) = 0 \}$

The $g^{(\alpha)}$-module $L(\Lambda)$ is completely reducible. $\quad L(\Lambda)|_{g(\alpha)}$ for $\alpha \in \Delta_+^{im}$

$\| \ n_-^{(\alpha)} \oplus (\mathbb{C}\alpha^\vee(\alpha)) \oplus n_+^{(\alpha)}$

Fact 4: Comparing Cor 12.8(b) with (12.8.12)

We obtain the following interpretation of braching functions:

Modular function

$$\boxed{b_\lambda^{\hat{\lambda}}(\dot{g}) = \mathrm{tr}_{U(\lambda,\lambda)}\, q^{L_0^{g.\dot{g}} - \frac{1}{24}(c(k) - \dot{c}(\dot{k}))}} \qquad q = e^{-\delta}$$

Cor 12.8(b) $\boxed{L_0 = h_\lambda I - d}$ $= \frac{T_0}{2(k+h^\vee)}$ If V is the g-module $L(\lambda)$

(12.8.12) $m_\lambda = h_\lambda - \frac{1}{24}c(k)$ $\quad \dot{m}_\lambda = \dot{h}_\lambda - \frac{1}{24}c(\dot{k})$ $\quad \hat{L}(\dot{g}_i)$

$\cancel{L_0^{g.\dot{g}} = -d}$ $\qquad \cancel{L_0^{g} = \left(\sum_i \frac{\Omega_i}{2(k_i + h_i^\vee)} - d\right)}$

$\cancel{L_0^{\dot{g}} = \left(\sum_i \frac{\Omega_i}{2(k_i + h_i^\vee)} - d\right)}$

$\cancel{e^{-\delta\left(d + \frac{c(k) - c(\dot{k})}{24}\right)} \sum_{n \in \mathbb{Z}} \mathrm{mult}_\lambda(\lambda - n\delta, \dot{g})}$

$\parallel$

$e^{-(m_\lambda - \dot{m}_\lambda)\delta} \sum_{m \in \mathbb{Z}} \mathrm{mult}_\lambda(\lambda - n\delta, \dot{g})\, e^{-n\delta}$

$\cancel{= q^{-\frac{c(k) - c(\dot{k})}{24}}\, \mathrm{tr}_{U(\lambda,\lambda)}\, q^{L_0^{g,\dot{g}}}}$? $\cancel{\text{for } \lambda \in \max(\lambda)}$

$\Omega_{(i)}(v) = (\lambda_{(i)} + 2\rho \mid \lambda_{(i)})\, v$ if v is singular of weight λ

$\Omega_{(i)}$ commutes with g $\qquad L_0^{g} = \sum_i \frac{\Omega_i}{2(k_i + h_i^\vee)} - d$

Cor 12.8(b) $L_0^{g} = h_\lambda I - d$ on $L(\lambda)$, $(\lambda \in P_+)$

$m_\lambda = h_\lambda - \frac{1}{24}c(k)$ $\qquad$ 特征值 $h_\lambda - \dot{h}_\lambda$

$\chi_\lambda = e^{-m_\lambda\delta}\, \mathrm{ch}\, L(\lambda) = A_{\lambda+\rho}/A_\rho$ $\qquad L_0^{g\dot{g}}$

$$b_\lambda^{\hat{\lambda}}(\dot{g}) = e^{-(m_\lambda - \dot{m}_\lambda)\delta} \sum_{n \in \mathbb{Z}} \mathrm{mult}_\lambda(\lambda - n\delta, \dot{g})\, e^{-n\delta}$$

$$m_\lambda - \dot{m}_\lambda = (h_\lambda - \dot{h}_\lambda) - \frac{1}{24}(c(k) - c(\dot{k}))$$

$$= q^{(h_\lambda - \dot{h}_\lambda) - \frac{1}{24}(c(k) - \dot{c}(\dot{k}))} \sum_{n \in \mathbb{Z}} \mathrm{mult}_\lambda(\lambda - n\delta, \dot{g})\, e^{-n\delta}$$

$$= q^{-\frac{1}{24}(c(k) - \dot{c}(\dot{k}))}\, q^{(h_\lambda - \dot{h}_\lambda)} \sum_{n \in \mathbb{Z}} \mathrm{mult}_\lambda(\lambda - n\delta, \dot{g})\, q^{n}$$

$$= q^{-\frac{1}{24}(c(k) - \dot{c}(\dot{k}))}\, \mathrm{tr}_{U(\lambda,\lambda)}\, q^{L_0^{g\dot{g}}} \quad \text{for } \lambda \in \max(\lambda; \dot{g})$$

By (12.12.3) $L(\Lambda) = \bigoplus_{\lambda \in \dot{P}_+^k \bmod \epsilon \dot{\delta}} \dot{L}(\lambda) \otimes U(\Lambda,\lambda)$

$U(\Lambda,\lambda) = \bigoplus_{n \in \mathbb{Z}} L(\Lambda)_{\lambda - n\delta}^{n+}$

$\chi_{U(\Lambda,\lambda)} = e^{(m_\Lambda - \dot{m}_\lambda)\delta} ch(U(\Lambda,\lambda))$

which immediately implies an equation for normalized characters,

$= e^{-(m_\Lambda - \dot{m}_\lambda)\delta} \sum_{n \in \mathbb{Z}} (\lambda - n\delta, \dot{g}) e^{-n\delta}$

$$(12.12.4) \quad \boxed{\chi_\Lambda = \sum_{\lambda \in \dot{P}_+^k \bmod \epsilon\delta} \dot{\chi}_\lambda \, b_\lambda^\Lambda(\dot{g})}$$

$= b_\lambda^\Lambda(\dot{g})$

Now, we can prove the following important prop.

prop 12.12 (a) The module $L(\Lambda)$, $\Lambda \in P_+$ viewed as a coset Vir-module

$L(\Lambda) \to$ decompose into an orthogonal direct of sum of unitarizable irreducible h.w.m.

The $\mathfrak{g}$-module $L(\Lambda)$ carries a positive definite Hermitian form $\langle\cdot|\cdot\rangle$ with respect to

Pf: By prop 12.8 $\longrightarrow$ $(T_n), (T_{-n})$ are adjoint with respect the contravariant

$(x^{(n)})^* = w(x)^{(-n)}$, where w is the compact Hermitian form $H(T_n(x)|y) = -H(x|(T_n)^*(y))$

the coset Vir-module $L(\Lambda)$, $\Lambda \in P_+$ is unitary

antilinear anti-involution of $\mathfrak{g} \leftarrow (\cdot|\cdot)$, choose $\{v_j\}$ of the fixed point set of $-w$ (the compact form of $\mathfrak{g}$)

Hence all $U(\Lambda,\lambda)$ are unitary. s.t $(v_j|v_j) = \delta_{j,k}$ and put $u_j = u^j = i v_j$ and d (a basis)

Also, all eigenspaces of $L_0^{\mathfrak{g},\dot{\mathfrak{g}}}$ on $U(\Lambda,\lambda)$ are finite dimensional and also for $L_n^{\dot{g}}$

the we see that $(L_n^{\mathfrak{g}})^* = L_{-n}^{\mathfrak{g}}$

its spectrum is bounded below

Hence the coset Vir-module $L(\lambda)$ is unitary.

(a) follow from (12.2.3) and prop 11.12(c) In particular, it is complete reducible.

12.12.3

$\varphi(q) = \prod_{n=1}^{\infty} (1-q^n)$ $\quad L(\Lambda) = \bigoplus \dot{L}(\lambda) \otimes U(\Lambda,\lambda)$

Prop 11.12 (c) V be a unitarizable Vir-module s.t. d_0 is diagonalizable with finit dimensional eigenspaces and with spectrum bounded below.

Then V decomposes into an orthogonal direct sum of unitarizable Vir-modules $L(c,h)$ and the spectrum of d_0 is non-negative.

(b). If $\Lambda \in P_+$ and $\mathrm{mult}_\Lambda(\lambda;\dot{g}) \neq 0$, then $h_\Lambda \geq \dot{h}_\lambda$

(c). If $k_0 \geq 0$ and $k_i \in \mathbb{Z}_+$ for $i > 0$, then $c(k) \geq \dot{c}(k)$

$\left[c(k) = \frac{k(\dim \mathfrak{g})}{k + h^\vee} \quad \dot{c}(k) = \frac{k(\dim \dot{g})}{k + \dot{h}^\vee} \right]$

Pf: Let $v \in L(\Lambda)$ be a highest-weight vector of a $\hat{L}(\dot{g})$-submodule $\dot{L}(\lambda)$ of $L(\Lambda)$

Using Cor 12.8b we obtain: $\longrightarrow$ $\boxed{\text{If } V = L(\Lambda), \text{ then } L_0 = h_\Lambda I - d}$

$L_0^{\mathfrak{g}\dot{\mathfrak{g}}}(v) = (h_\Lambda - \dot{h}_\lambda) v$

chapter 2

$\left(L_0^{\dot{g}} = \sum_i \frac{\Omega_i}{2(k_i + h_i^\vee)} - d \right.$, $\quad \Omega_i(v) = (\lambda_{(i)} + 2\rho | \lambda_{(i)}) v \quad L_0^{\mathfrak{g}} = \sum_i \frac{\Omega_i(v)}{2(k_i + h_i^\vee)} - d(v)$

$= \sum_i \frac{(\lambda_{(i)} + 2\rho | \lambda_{(i)}) v}{2(k_i + h_i^\vee)} - \lambda(d) v = \left(\sum_i h_{\lambda(i)} - \lambda(d) \right) v = (h_\lambda - \lambda(d)) v$

$\Rightarrow L_0^{\mathfrak{g}\dot{\mathfrak{g}}}(v) = (h_\Lambda - \dot{h}_\lambda) v \rightarrow L_0^{\mathfrak{g}\dot{g}}$ 对于 $L_0^{\mathfrak{g}} = (h_\Lambda - d)$ on $L(\Lambda)$ $(\Lambda \in P_+)$

$\boxed{6}$

$\Rightarrow L_0^{gg'}(v) = (h_\Lambda - h_\lambda)v \longrightarrow$ 对应 d_0 对角

(b) (c) follow from prop 11.12 $\to$ (a) Vir mod $L(c,h)$ is unitarizable, then $h \geq 0$, and $c \geq 0$

(b) If $V = L(0,h)$ is unitarizable, then $h=0$, and hence V is the trivial 1-dimensional Vir-module

(c). V be a unitarizable Vir-mod s.t d_0 is diagonalizable with ~~finite-dimensional~~ ~~eigenvo~~ eigenspaces and with spectrum bound below. Then V decompose into an orthogonal direct sum of unitarizable Vir-module $L(c,h)$, and the spectrum of d_0 is non-negative

The fact that $h_\Lambda - h_\lambda$ is the ~~minimal~~ eigenvalue of $L_0^{g,g'}$ on $U(\Lambda,\lambda)$ ~~equal~~ $\downarrow$ unitary

~~to ha-h~~ ($L(\Lambda)$ is unitary and completely reducible)

$\Rightarrow h_\Lambda - h_\lambda \geq 0$ $\qquad d_0 \to L_0^{gg'}$

(c) $c(k) = \sum_i c(k_i)$

$\hat{c}(k) = \sum_i \hat{c}(k_i)$

By prop 11.2

$L(c,h)$ is unitarizable.

$h \geq 0$ $\quad c \geq 0$

i.e $c(k) - \hat{c}(k) \geq 0$

given two real number c and h, there exists a unique irreducible Vir module $L(c,h)$

$v \in L(c,h)$ $\qquad d_j(v) = 0$ for $j > 0$

$d_0(v) = h \cdot v$

$C(v) = c \cdot v$
$\downarrow$ conformal anomaly

d_0 is diagonalizable.

$L(c,h) = \sum_{j \in \mathbb{Z}_+} L(c,h)_{h+j}$

$L(c,h)_\lambda$ deonte $d_0 \to \lambda$

$h \to$ is the minimal eigenvalue of d_0
$\downarrow$
Conformal dimensional dimenis of $L(c,h)$

If $\mathfrak{g}$ is semisimple, then the sum in (12.12.3) is clearly finite

$L(\lambda)$ with respect to $\hat{L}(\dot{\mathfrak{g}}) \oplus Vir \longrightarrow L(\lambda) = \bigoplus L(\lambda) \otimes U(\lambda,\lambda)$

l.t.p.f modules

in general not the case, For example $\mathfrak{g} = \bar{\mathfrak{h}}$

We shall transform this sum to a finite one using the same trick (方式) as in §12.7

For this we shall assume that $\dot{\mathfrak{g}}$ 由大到小 取定 $\quad \dot{\mathfrak{g}}_{(0)}$ 为 $\dot{\mathfrak{g}}$ 的中心

(12.12.5) $\quad \dot{\mathfrak{g}}_{(0)} \cap \bar{Q}^{\vee}$ spans $\dot{\mathfrak{g}}_{(0)}$ over $\mathbb{C}$,

where $\bar{Q}^{\vee} \subset \bar{\mathfrak{h}}$ is the coroot lattice of $\mathfrak{g}$. Introduce the lattice:

$\dot{M}_0 = \nu^{-1}(\dot{\mathfrak{g}}_{(0)} \cap \bar{Q}^{\vee})$, which is a sublattice of the lattice $M \hookrightarrow \mathfrak{g} \hookrightarrow \dot{\mathfrak{g}}_{(0)}$

. Let $\dot{\mathfrak{h}}' = \bigoplus_{i \geq 1} \dot{\mathfrak{h}}_{(i)}$, Then we have: $\underset{\mathfrak{h} \cap \hat{L}(\dot{\mathfrak{g}})}{\overset{}{\dot{\mathfrak{h}}}} = \dot{\mathfrak{h}}_{(0)} + \dot{\mathfrak{h}}'$, $\dot{\mathfrak{h}}_{(0)} \cap \dot{\mathfrak{h}}' = \mathbb{C}d$

为 $\dot{\mathfrak{g}}$ 的子代数

$\dot{\mathfrak{g}}_{(0)}$ 为李代数 $\dot{\mathfrak{g}}$ 的 Center

$\to$ ~~每Center~~ 每个 $\mathfrak{g}_{(i)}$ 里都有 d?

$\mathfrak{g}$ reductive f.d. Liealgebra

$\dot{\mathfrak{g}}$ reductive subalg.

(1) $|\dot{\mathfrak{g}}$ non-dege.

d 不是在loop实现里才有?

$$L_0^{\mathfrak{g}} = \sum_i \frac{\Omega_i}{2(k_i + h_i^{\vee})} - \underline{d}$$

$\mathfrak{g} = \bigoplus_{i \geq 0} \mathfrak{g}_{(i)}$

$\dot{\mathfrak{g}} = \bigoplus_{i \geq 0} \dot{\mathfrak{g}}_{(i)}$

$k = (k_0 + k_1 \cdots k_n \cdots)$ $\quad k_i = \Lambda(K_i), h = \Lambda(k_i)$

$c(k) = \sum_i c(k_i)$

$h_\lambda = \sum_j h_{\lambda j}$ $\quad \dot{\mathfrak{h}}_{(i)} = \mathfrak{h} \cap \hat{L}(\dot{\mathfrak{g}}_{(i)})$

$m_\lambda = \sum_j m_{\Lambda(j)}$

$\mathfrak{g} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$

$\dot{\mathfrak{g}} = \dot{\mathfrak{n}}_- \oplus \dot{\mathfrak{h}} \oplus \dot{\mathfrak{n}}_+$

$\underset{\hat{L}(\mathfrak{g})}{\mathfrak{h}} = \bar{\mathfrak{h}} + c + \mathbb{C}d$

hence $\dot{\mathfrak{h}}^{+} = \dot{\mathfrak{h}}_{(0)} + \dot{\mathfrak{h}}'^{+}$ and $\dot{\mathfrak{h}}_{(0)}^{+} \cap \dot{\mathfrak{h}}'^{+} = \mathbb{C}\delta$

Given $\lambda \in \dot{\mathfrak{g}}^*$, we have a decomposition :

(12.2.6) $\lambda = \lambda_{(0)} + \lambda^{(1)}$, where $\lambda_{(0)} \in \dot{\mathfrak{g}}^*_{(0)}$, $\lambda^{(1)} \in \dot{\mathfrak{g}}'^*$

which is unique up to adding multiples of $\dot{\delta}$

Due to (12.9.3) and ~~(12.8.3)~~ (12.8.13) we have for $\lambda \in \dot{P}^k_+$

$\chi_\Lambda = e^{-m_\Lambda \delta} \mathrm{ch} L(\Lambda) = \prod_i \chi_{\Lambda(i)}$

$\dot{\mathfrak{g}}$ abelian $\mathfrak{g}$-mod $L(\Lambda)$. $\Lambda \in \mathfrak{h}^*$. $k \neq 0$ $\mathrm{ch} L(\Lambda) = e^{\hat{\Lambda}} / \varphi(e^{-\delta})^{\dim \dot{\mathfrak{g}}}$

$\chi_\Lambda = e^{-m_\Lambda \delta} \mathrm{ch} L(\Lambda) = e^{-\frac{|\Lambda|^2}{2k}\delta + \Lambda} / \eta^{\dim \dot{\mathfrak{g}}}$

$\dot{\mathfrak{g}}_{(0)}$ $\hat{L}(\mathfrak{g}_{(0)})$-module $\dot{L}(\lambda_0)$

(12.12.7) $\dot{\chi}_\lambda = \dot{\chi}_{\lambda^{(1)}} \cdot \dot{\chi}_{\lambda_{(0)}} = \dot{\chi}_{\lambda^{(1)}} \left(e^{-\frac{|\lambda_{(0)}|^2}{2k}\delta + \lambda_{(0)}} / \eta^{\dim \dot{\mathfrak{g}}_{(0)}} \right)$

Here $\dot{\chi}_{\lambda^{(1)}} = \prod_{j \geq 1} \dot{\chi}_{\lambda(j)}$ is normalized character of $\hat{L}(\dot{\mathfrak{g}}')$-module $\dot{L}(\lambda^{(1)})$,

where $\dot{\mathfrak{g}}'$ is the derived algebra of $\dot{\mathfrak{g}}$

It is clear that for $\alpha \in \dot{M}_0$, we have

$t_\alpha(\dot{\chi}_{\lambda^{(1)}}) = \dot{\chi}_{\lambda^{(1)}}$ and $b^{\hat{}}_{t_\alpha(\lambda)} = b^{\hat{}}_\lambda$ (braching function 在 t_α $\alpha \in \dot{M}$ 作用下不变).

Using this, we rewrite (12.12.3) in the following form: $L(\Lambda) = \bigoplus_{\lambda \in \dot{P}^k_+ \bmod \mathbb{C}\delta} \dot{L}(\lambda) \otimes U(\Lambda, \lambda)$

应该是 (12.12.4) $\chi_\Lambda = \sum_{\lambda \in \dot{P}^k_+ \bmod \mathbb{C}\delta} \dot{\chi}_\lambda b^{\hat{}}_\lambda(\dot{\mathfrak{g}})$

(12.12.8)

$$\chi_\Lambda = \sum_{\substack{\lambda \in \dot{P}^k_+ \bmod \mathbb{C}\delta \\ \lambda \bmod k_0 \dot{M}_0}} b^{\hat{}}_\lambda(\dot{\mathfrak{g}}) \dot{\chi}_{\lambda^{(1)}} \underbrace{\left(\sum_{\alpha \in \dot{M}_0} e^{t_{(\alpha)}(\lambda)} \cdot e^{-\frac{|\lambda_0|^2}{2k_0}\delta} / \eta^{l_0} \right)}_{\dot{\chi}_{\lambda_{(0)}}}$$

$$= \sum b^{\hat{}}_\lambda(\dot{\mathfrak{g}}) \dot{\chi}^{(1)}_\lambda \left(\dot{\Theta}^{\circ}_{\lambda_0} / \eta^{l_0} \right)$$

where $\dot{\Theta}^{\circ}_{\lambda_0} = e^{-\frac{|\lambda_{(0)}|^2}{2k_0}\delta} \sum_{\alpha \in \dot{M}_0} e^{t_{(\alpha)}(\lambda)}$ is the theta function associated to the lattice $\dot{M}_0$

prop 11.13, $\Lambda \in P^1_+$, for $\chi^{(1)}_\Lambda$, $X = A, D, E$

$a^{\hat{}}_\Lambda = \prod_{n=1}^{\infty} (1 - e^{-n\delta})^{\mathrm{mult}\, n\delta}$

$e^{-\frac{1}{2}|\Lambda|^2\delta} \mathrm{ch} L(\Lambda) = a^{\hat{}}_\Lambda \sum_{\gamma \in \bar{Q} + \bar{\Lambda}} e^{\Lambda_0 + \gamma - \frac{1}{2}|\gamma|^2\delta}$ then by Lema 12.7

$\Rightarrow e^{-\frac{1}{2}|\Lambda|^2\delta} \mathrm{ch} L(\Lambda) = \sum_{\gamma \in \bar{Q} + \bar{\Lambda}} e^{\Lambda_0 + \gamma - \frac{1}{2}|\gamma|^2\delta} / \prod_{n=1}^{\infty} (1 - e^{n\delta})^{\mathrm{mult}\, n\delta}$

Rmk: (1) the sum on the right (12.12.8) is finite.

(2) (12.12.8) $\dot{\Theta}_{\lambda_0}$ is a generation of the theta function identity (12.7.3).

$h = h_\Lambda - h_\lambda = 0$

$\Theta_\lambda = e^{k\Lambda_0} \sum_{\gamma \in M + \bar{k}^{-1}\bar{\lambda}} e^{-\frac{1}{2}k|\gamma|^2\delta + k\gamma}$

$n^{\gamma}_\lambda = \frac{2(\rho|\lambda)}{(\Lambda|\Lambda)}$

§12.13. $k = 1 \Rightarrow a^{\hat{}}_\Lambda = ?$ $c(1) - \dot{c}(1) = 0$ then by prop 11.12(b) and (c)

the coset Vir-mod $U(\Lambda, \lambda)$ is trivial and $\dim U(\Lambda, \lambda) = 1$ and $b^{\hat{}}_\lambda(\dot{\mathfrak{g}}) = 1$